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FRACTIONAL DISCRETE-TIME COMPARTMENTAL LINEAR SYSTEMS

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Abstract: This paper introduces a class of fractional discrete-time compartmental linear systems. The fundamental system properties, including controllability and observability, are analysed. Furthermore, the eigenvalue assignment problem related with this class of systems is addressed. Theoretical considerations are demonstrated through a numerical example.

Key words: discrete-time system, fractional, compartmental, observability, controllability

1. INTRODUCTION

Fractional calculus is an extension of classical integer-order calculus that involves derivatives and integrals of non-integer (fractional) orders. The mathematical foundations of fractional calculus are presented in various monographs, such as [12], [13], and [14]. The applications of fractional calculus across various fields of science and engineering have attracted considerable attention in recent years. It has been used in areas such as mechanics, electrical engineering, biology, chemistry, and signal processing [13, 16, 18]. The fractional-order modeling of real-world phenomena are often more accurate than classical integer-order models. The theory of fractional systems is an expanding field that explores properties of systems, including stability, controllability, observability, realisability, and more [1, 2, 4, 7, 11, 15, 17, 19]. Standard and positive fractional linear systems have been discussed in monographs [8] and [10], respectively. A dynamical system is termed positive when its state variables and outputs take nonnegative values for any nonnegative inputs. Numerous models exhibiting positive behaviour can be found across fields such as engineering, biology, medicine, and economics. A comprehensive overview of research in positive systems theory is provided in [3, 6].

In the modelling process, compartmental linear systems are frequently used. These systems consist of separate compartments that are interconnected, each representing a subsystem containing a specific material. The transfer of material between compartments is governed by linear equations [5]. The fractional continuous-time compartmental systems have been studied in [9].

In this paper, fractional discrete-time compartmental time-invariant linear systems are introduced and analysed. To the best of the authors' knowledge, the problems of controllability, observability, and eigenvalue assignment have not yet been addressed for fractional discrete-time linear systems. This paper extends the fractional-order systems theory to this concern. A key advantage of discrete-time fractional-order numerical models is their ability to describe complex dynamical systems with non-local, memory-based interactions, providing more accurate and nuanced representations. These models are suitable for real-world processes where the future state depends not only on the current value but also on the entire history of the system.

The structure of the paper is as follows. Section 2 provides the fundamental definitions and theorems related to fractional and positive linear systems. In Section 3, the concept of fractional discrete-time compartmental linear systems is introduced. Section 4 is devoted to the analysis of controllability and observability of the proposed systems, while Section 5 addresses the eigenvalue assignment problem. Finally, concluding remarks are presented in Section 6.

The notation used in this paper is as follows: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, Z_+ - the set of nonnegative integers, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, I_n- the $n\times n$ identity matrix.

2. STANDARD LINEAR DISCRETE-TIME SYSTEMS

Let us consider a linear discrete-time system represented by the following equations.

$$x_{i+1}=Ax_i+Bu_i,\ i\in Z_+=\{0,\ 1,\ \dots\},\tag{2.1a}$$

$$y_i=Cx_i,\tag{2.1b}$$

with the initial condition x_0, where $x_i\in\Re^n$ represents the state vector, $u_i\in\Re^m$ the control input, and $y_i\in\Re^p$ the system output, while $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$ and $C\in\Re^{p\times n}$ are the corresponding system matrices.

Definition 2.1. [6, 8] The linear system (2.1) is called (internally) positive if $x_i\in\Re_+^n$ and $y_i\in\Re_+^p$, $i\in Z_+$ for any initial conditions $x_0\in\Re_+^n$ and all inputs $u_i\in\Re_+^m$, $i\in Z_+$.

Theorem 2.1. [6, 8] The linear system (2.1) is positive if and only if:

$$A\in\Re_+^{n\times n},\ B\in\Re_+^{n\times m},\ C\in\Re_+^{p\times n}\tag{2.2}$$

Definition 2.2. The linear system (2.1) is called asymptotically stable if $\lim_{i\to\infty}x_i=0$ for $u_i=0$ and any initial $x_0\in\Re^n$.

Theorem 2.2. [6, 8] The linear system (2.1) is asymptotically stable if the matrix A is a Schur matrix.

Theorem 2.3. [6, 8] The positive linear system (2.1) is asymptotically stable if and only if:
1. all coefficients of the polynomial

$$p_A(z) = det[I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \quad (2.3)$$

are positive, i.e., $a_i > 0$ for $i = 0,1,\ldots,n-1$.
2. there exists strictly positive vector

$\lambda^T = [\lambda_1 \quad \cdots \quad \lambda_n]^T$, $\lambda_k > 0$, $k = 1,\ldots,n$ such that

$$A\lambda < 0 \text{ or } \lambda^T A < 0. \quad (2.4)$$

Let us now examine a linear fractional discrete-time system given by the following equations:

$$\Delta^\alpha x_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ = \{0,\ 1,\ \ldots\}, \quad 0 < \alpha < 1 \quad (2.5a)$$

where $x_i \in \Re^n$, $u_i \in \Re^m$ and $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ and

$$\Delta^\alpha x_i = \sum_{j=0}^{i}(-1)^j \binom{\alpha}{j} x_{i-j}$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j = 0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for} \quad j = 1,2,\ldots \end{cases} \quad (2.5b)$$

is the fractional α-order difference of x_i.

Substitution of (2.5b) into (2.5a) yields

$$x_{i+1} = A_\alpha x_i - \sum_{j=2}^{i+1} c_j x_{i-j+1} + Bu_i,\ i \in Z_+, \quad (2.6a)$$

where

$$A_\alpha = A + I_n\alpha,\ c_j = (-1)^{j+1}\binom{\alpha}{j},\ j = 1,2,\ldots. \quad (2.6b)$$

Definition 2.3. [6, 8] The fractional system (2.5) is called (internally) positive if $x_i \in \Re_+^n$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$.

Theorem 2.4. [6, 8] The fractional system (2.5) is positive if and only if

$$A_\alpha \in \Re_+^{n\times n}. \quad (2.7)$$

Definition 2.4. [6, 8] The fractional positive system (2.5) is called asymptotically stable if

$$\lim_{i\to\infty} x_i = 0 \text{ for all } x_0 \in \Re_+^n. \quad (2.8)$$

Theorem 2.5. [6, 8] The fractional positive system (2.5) is asymptotically stable if and only if one of the equivalent conditions is satisfied:
1. all coefficients of the polynomial

$$p_A(z) = det[I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \quad (2.9)$$

are positive, i.e., $a_i > 0$ for $i = 0,1,\ldots,n-1$.
2. there exists strictly positive vector

$\lambda^T = [\lambda_1 \quad \cdots \quad \lambda_n]^T$, $\lambda_k > 0$, $k = 1,\ldots,n$ such that

$$[A - I_n]\lambda < 0 \text{ or } \lambda^T[A - I_n] < 0. \quad (2.10)$$

3. STATE EQUATIONS OF THE FRACTIONAL DISCRETE-TIME LINEAR COMPARTMENTAL SYSTEMS

Let us consider the compartmental discrete-time time invariant system consisting of n compartments (Fig.1).

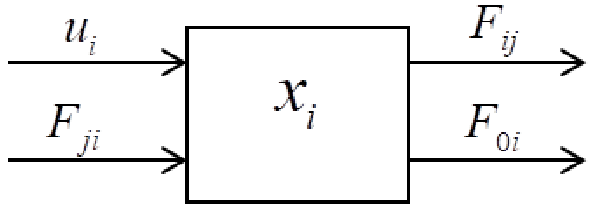

Fig. 1. The i -th subsystem of the compartmental system

Let: $x_i = x_i(k)$, $i = 1,\ldots,n$ be the amount of a material of the i -th compartmental at the time instant k,
$F_{ij}(k) > 0$ be the output flow of the material from the j -th to the i -th compartmental ($i \neq j$), between the k -th and $k+1$-th time instants,
$F_{0i}(k) > 0$ be the output of the material from the i -th ($i = 1,\ldots,n$) compartmental to the environment,
$u_i = u_i(k)$ be the output flow of the material to the i -th compartmental from environment.

It is assumed that the input material is instantaneously mixed with the material already present in the compartment and that $F_{ij}(k)$ depends linearly on $x(k)$, i.e.,

$$F_{ij}(k) = f_{ij}x_j(k) \quad \text{for} \quad i \neq j,\ \ i = 1,\ldots,n,\ \ j = 1,\ldots,n, \quad (3.1)$$

where f_{ij} is a coefficient depending on $x_j(k)$ and the discrete-time instant k.

The system is linear if f_{ij} is independent of $x_j(k)$ and it is additionally time-invariant if f_{ij} is independent of k.

From the balance of material of the i -th compartment we have the following fractional difference equation

$$\Delta^\alpha x_i(k+1) = \sum_{\substack{i=1 \\ i\neq j}}^{n} f_{ij}x_j(k) + f_{ii}x_i(k) + u_i(k) \quad \text{for} \quad i = 1,\ldots,n, \quad (3.2)$$

where $\Delta^\alpha x_i$ is defined by (2.5b) and $x_i(k)$ denotes the amount of material in the i -th compartment at time step k, i.e.,

$$f_{ii}x_i(k) = x_i(k) - f_{0i}x_i(k) - \sum_{\substack{i=1 \\ i\neq j}}^{n} f_{ij}x_i(k) = \left(1 - f_{0i} - \sum_{\substack{i=1 \\ i\neq j}}^{n} f_{ij}\right)x_i(k). \quad (3.3)$$

From equation (3.3) we have

$$f_{ii} = 1 - f_{0i} - \sum_{\substack{i=1 \\ i\neq j}}^{n} f_{ij} \text{ for } i = 1,\ldots,n. \quad (3.4)$$

Note that if $u_j(k) = 0$, then the output flow of material from the j -th compartment at the time instant $k+1$ cannot exceed the total amount of material present in the compartment at time instant k, i.e.,

$$\sum_{i=1}^{n} f_{ij} \leq 1 \text{ for } j = 1,\ldots,n \text{ and } f_{ij} \geq 0. \quad (3.5)$$

Definition 3.1. The matrix $F \in \Re_+^{n\times n}$ satisfying the condition (3.5) is called the compartmental matrix of the fractional discrete-time linear system.

Using (3.3) for $i = 1,\ldots,n$ we obtain the state equation of the compartmental system in the form

$$x(k+1) = Fx(k) + Bu(k),\ i = 1,\ldots,n \quad (3.6a)$$

where

$$x(k)=\begin{bmatrix}x_1(k)\\ \vdots\\ x_n(k)\end{bmatrix},\quad u(k)=\begin{bmatrix}u_1(k)\\ \vdots\\ u_n(k)\end{bmatrix},\quad F=\begin{bmatrix}f_{11} & \dots & f_{1n}\\ \vdots & \dots & \vdots\\ f_{n1} & \dots & f_{nn}\end{bmatrix}. \tag{3.6b}$$

The output equation of the compartmental system has the form

$$y(k)=Cx(k), \tag{3.6c}$$

where $C\in\Re_+^{p\times n}$.

From (3.6) it follows that the fractional compartmental systems are positive linear systems.

4. CONTROLLABILITY AND OBSERVABILITY OF STANDARD AND COMPARMENTAL LINEAR SYSTEMS

Let us consider a linear discrete-time system described by the following equations:

$$x_{i+1}=Ax_i+Bu_i,\ i\in Z_+=\{0,\ 1,\ \dots\}, \tag{4.1a}$$

$$y_i=Cx_i, \tag{4.1b}$$

with the initial condition x_0, where $x_i\in\Re^n$, $u_i\in\Re^m$ and $y_i\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$ and $C\in\Re^{p\times n}$ are system matrices.

Definition 4.1. The linear system (4.1) (or the pair $(A,\ B)$) is called controllable in the interval time $[0,\ i_f]=0,1,\dots,i_f$ if the exists an input u_i for $i\in[0,\ i_f]$ which steers the state of the system from initial state $x_0\in\Re^n$ to the given final state x_f, i.e., $x_{i_f}=x_f$.

Theorem 4.1. The linear system (4.1) is controllable if and only if one of the following conditions is satisfied:

1. $\operatorname{rank}[B\quad AB\quad \dots\quad A^{n-1}B]=n$ (4.2)
2. $\operatorname{rank}[I_nz-A\quad B]=n$ for all $z\in$ C, (4.3)

where C is the field of complex numbers.

Definition 4.2. The linear system (4.1), or equivalently the pair $(A,\ C)$, is called observable if it is possible to uniquely determine the initial state x_0 based on the input u_i and output y_i for $i=0,1,\dots,i_f$.

Theorem 4.2. The linear system (4.1) is observable if and only if at least one of the following conditions is satisfied:

1. $\operatorname{rank}\begin{bmatrix}C\\ \mathrm{CA}\\ \vdots\\ CA^{n-1}\end{bmatrix}=n,$ (4.4)
2. $\operatorname{rank}\begin{bmatrix}I_nz-A\\ C\end{bmatrix}=n$, for all $z\in$ C, (4.5)

Now let us consider the fractional compartmental linear system (3.6)

Definition 4.3. The fractional compartmental linear system (3.6), or equivalently the pair $(F,\ B)$, is called reachable on the time interval $[0,\ i_f]$ if there exists an input sequence u_i for $i\in[0,\ i_f]$ which steers the system state from the zero initial condition to a given final state x_f, i.e., $x_{i_f}=x_f$.

A matrix $F\in\Re^{n\times n}$ is called monomial if each of its rows and each of its columns contains exactly one positive entry, and all other entries are zero.

Theorem 4.4. The fractional compartmental linear system (3.6) is reachable if the matrix

$$R_f=\sum_{i=0}^{n-1}F^i(F^T)^i \tag{4.6}$$

is monomial. The input which steers the system state to $x_{i_f}=x_f$ is given by

$$u_i=(F^T)^{i_f-i-1}{R_f}^{-1}x_f \tag{4.7}$$

Proof. When matrix (4.6) is monomial, its inverse $R_f^{-1}\in\Re_+^{n\times n}$ is nonnegative matrix. Consequently, the input (4.7) is also nonnegative. Given that $x_0=0$, applying (4.7) yields

$$x_f=\sum_{i=0}^{i_f-1}F^{i_f-i-1}Bu_i=\sum_{i=0}^{i_f-1}F^{i_f-i-1}(F^T)^{i_f-i-1}{R_f}^{-1}x_f=x_f \tag{4.8}$$

since $B=I_n$.

Therefore, the input (4.7) steers the state of the system from x_0 to $x_{i_f}=x_f$. □

Theorem 4.5. The fractional compartmental positive linear system (3.6) is reachable in time $[0,\ i_f]$ if and only if the matrix $F\in M_n$ is monomial.

Proof. Sufficiency. If $F\in M_n$ is monomial then $F^i\in\Re_+^{n\times n}$ is also monomial. In this case the matrix

$$R_f=\sum_{i=0}^{i_f-1}F^{i_f-i-1}(F^T)^{i_f-i-1}=\sum_{i=0}^{i_f-1}F^i(F^T)^i \tag{4.9}$$

is monomial.

Necessity. From Cayley-Hamilton theorem [6] we have

$$F^i=\sum_{j=0}^{m-1}a_{ij}F^j,\ i=m,m+1,\dots \tag{4.10}$$

where $a_{ij}$are some nonzero real coefficients.

Using (4.10) we obtain

$$x_f=[B\quad FB\quad \dots\quad F^{n-1}B]\begin{bmatrix}v_{0_f}\\ v_{1_f}\\ \vdots\\ v_{n-1,f}\end{bmatrix} \tag{4.11a}$$

where

$$v_{i_f}=\sum_{i=0}^{i_f}a_{ij}u_i. \tag{4.11b}$$

Therefore, for given $x_f\in\Re_+^n$ it is possible to find the nonnegative v_{i_f} for $i=0,1,\dots,n-1$ if and only if

$$\operatorname{rank}[B\quad FB\quad \dots\quad F^{n-1}B]=n. \tag{4.12}$$

Observe that for the nonnegative system defined in (4.11b), a nonnegative input $u_i\in\Re_+^m$ can be determined. Hence, the proof is complete. □

Observability of fractional positive compartmental linear systems is defined analogously to that in standard positive linear systems. Since it is determined exclusively by the matrices A and C, and not by B. Consequently, system (2.1) is replaced by the fractional positive compartmental linear system of the following form

$$\Delta^\alpha x_{i+1}=Fx_i,\ ,\ i\in Z_+=0,1,2,\dots,\ 0<\alpha<1, \tag{4.13a}$$

$$y=Cx_i, \tag{4.13b}$$

where $x_i\in\Re^n$, $y_i\in\Re^p$ and $C\in\Re_+^{p\times n}$.

The solution to the equation (4.13a) with (2.6b) has the form

$$x_i=\Phi_ix_0+\sum_{j=0}^{i-1}\Phi_{i-j-1}Bu_j, \tag{4.14a}$$

where

$$\Phi_{i+1} = (F + I_n\alpha)\Phi_i + \sum_{j=2}^{i+1}(-1)^{j+1}\binom{\alpha}{i}\Phi_{i-j+1}, \quad (4.14b)$$

with $\Phi_0 = I_n$.

Definition 4.4. The fractional positive compartmental linear system (4.13) is called observable on the interval $[0,\ i_f]$ if knowledge of the output y_i over the interval $[0,\ i_f]$ enables unique determination of the initial state x_0.

Theorem 4.6. The fractional positive compartmental linear system (4.13) is observable on the interval $[0,\ i_f]$ if and only if the matrix

$$\Phi^T C^T C\Phi \in \Re_+^{n\times n} \quad (4.15)$$

is monomial.

Proof. Substituting (4.14a) into (4.13b) we obtain

$$y = C\Phi x_0. \quad (4.16)$$

Note that $[\Phi^T C^T C\Phi]^{-1} \in \Re_+^{n\times n}$ if and only if the matrix (4.15) is monomial. Consequently, equation (4.16) yields

$$x_0 = [\Phi^T C^T C\Phi]^{-1}\Phi^T C^T y \in \Re_+^n \quad (4.17)$$

since $\Phi^T C^T y \in \Re_+^{n\times n}$ for y_i.

5. EIGENVALUE ASSIGNMENT IN THE STANDARD AND FRACTIONAL COMPARMENTAL LINEAR SYSTEMS

Let us consider the fractional compartmental system (3.3) under state feedback control

$$u = Kx, \quad (5.1)$$

where $K \in \Re^{n\times n}$.

Assuming $B = I_n$, it follows from equation (5.1) that

$$\Delta^\alpha x_{i+1} = F_c x_i, \quad (5.2)$$

where

$$F_c = F - K. \quad (5.3)$$

Based on the given matrix A and the desired close-loop matrix A_c from (5.3), the following expression can be derived

$$K = F - F_c. \quad (5.4)$$

Accordingly, the following theorem is established.

Theorem 5.1. Given the fractional compartmental system (3.3), there always exists a state feedback (5.1) such that the closed-loop system matrix F_c achieves a specified set of eigenvalues.

Example 5.1. The matrix F of the fractional compartmental linear system is given by

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix} \quad (5.5)$$

and its eigenvalues are: $z_1 = z_2 = 2$, $z_3 = -1$, since

$$det[I_3 z - F] = \begin{vmatrix} z & -1 & 0 \\ 0 & z & -1 \\ 4 & 0 & z-3 \end{vmatrix} = z^3 - 3z^2 + 4. \quad (5.6)$$

Determine the feedback matrix $K \in \Re^{3\times 3}$ such that the closed-loop system matrix F_c has eigenvalues: $\bar{z}_1 = -0.1$, $\bar{z}_2 = -0.2$, $\bar{z}_3 = -0.5$.

It should be noted that the desired closed-loop matrix F_c is not unique. Two alternative forms of F_c are considered below.

Case 1. The matrix F_c is assumed to be in the Frobenius canonical form, as in equation (5.5)

$$F_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.01 & -0.17 & -0.8 \end{bmatrix}. \quad (5.7)$$

In this case using (5.4), (5.5) and (5.6) we obtain

$$K = F - F_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -4.01 & -0.17 & 3.8 \end{bmatrix}. \quad (5.8)$$

Case 2. The matrix F_c F_c is assumed to be diagonal

$$F_c = \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}. \quad (5.9)$$

In this case we have

$$K = F - F_c = \begin{bmatrix} -0.1 & 1 & 0 \\ 0 & -0.2 & 1 \\ -4 & 0 & 3.5 \end{bmatrix}. \quad (5.10)$$

Note that the presented approach can be generalized to include output feedback strategies.

6. CONCLUDING REMARKS

Fractional, compartmental, time-invariant linear systems are analyzed, with a focus on their fundamental properties. Theoretical foundations, including key definitions and theorems related to standard and positive fractional linear systems, are outlined. A class of fractional compartmental discrete-time systems is introduced and studied. The concepts of controllability and observability are discussed for both standard and compartmental systems, followed by an examination of the eigenvalue assignment problem in the compartmental case. The results obtained may also be extended to descriptor discrete-time fractional linear systems.

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