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HEAT CONDUCTION PROBLEMS FOR HALF-SPACES WITH TRANSVERSAL ISOTROPIC GRADIENT COATING

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Abstract: The axisymmetric heat conduction problems of local surface heating for a FGM-coated body are considered. The following types of the coating were considered: a transversal isotropic gradient coating with continuously varying thermal properties; a coating consisting of a finite number of homogeneous transversal isotropic layers with slowly varying thermal properties; a coating consisting of a finite number of representative cells containing two homogeneous layers. The solution for a specially selected multilayer coating was compared with that for a transversal isotropic coating with exponentially varying properties. It has been shown that multilayer coatings with a step change of the thermal properties can be described using a homogenization method with microlocal parameters.

Key words: heat conduction problem, temperature, heat flux, transversal isotropic gradient coating, multi-layered structure

1. INTRODUCTION

In engineering practice, assessing the reliability of machine components or the strength of structures is closely related to the problems of heat generation and conduction. High temperature may cause an additional thermos-elastic deformation, thermal stress-induced cracking, and even a change of material properties. Therefore, the substrate is usually coated with a protective thermal barrier. The base material used in most high-temperature applications is ceramic. However, ceramics have some shortcomings, such as being brittle and susceptible to cracking. Protective coverings made of functionally graded material (FGM) are often used to alleviate these problems. Unlike a homogeneous or periodic coating, FGM- coating allows matching material properties at the interface. As a result, these coating structures are able to withstand high-temperature gradients without structural failures. Many experimental and numerical studies [1 – 3] have shown that FGMs used as coatings can effectively reduce thermal stresses, enhance interface bonding strength and improve the surface properties of materials.

Actually, FGMs are mixtures of two or more different materials. Volume fraction of each material varies along the thickness of the coating. The gradual change in material properties is adapted to meet the different requirements. The mathematical models of such materials are derived by using some averaging procedures. In the case of the gradient coating that has the properties of an isotropic solid at the macro level, classical averaging methods are still often used: the Voigt estimation [4] or the Reus's estimation [5]. However, these estimates are insufficient [6] to properly describe FGM-coating that that has the anisotropic properties at the macro level, in particular the multilayer structures. Therefore, increasing usage of multi-layered coatings requires development of more accurate mathematical models for analysis of experimental results.

The anisotropic properties of the multi-layer coating are taken into account in homogenization methods. Particularly noteworthy is the homogenization method with microlocal parameters [7, 8], which was previously used to solve problems for periodic multilayer media. In the problems of thermal conductivity for a multilayer coating with a periodic structure [9, 10], this method makes it possible to evaluate not only mean values of the analyzed state functions but also their local values in every layer of the periodicity cell.

It is usually assumed that the condition for applying the homogenization method is that a considered body consists of a periodically repeating representative unit cells. Kulchytsky-Zhyhailo et al. demonstrated [11, 12] the possibility of using the homogenization method with microlocal parameters for modeling cylindrical and spherical multilayer structures with slowly changing properties. The authors solved one-dimensional centrally-symmetric thermo-elasticity problems for a hollow multilayer cylindrical or spherical vessel. The vessel wall was made of a composite material assembled of concentric perfectly connected representative cells. A representative cell was composed of two homogeneous layers. The parameter describing the volumetric fraction of the first-kind layer in a representative cell may have been different for different representative cells. It has been shown that the solution based on the homogenization method is fully consistent with the solution for the problem in which each layer is considered separately.

In the case where a multilayer coating with a periodic structure is described by the homogenization method, the problem consists in solving the differential equations with constant coefficients. The analytical solution of obtained equations is known. A homogenized multilayer coating with slowly changing properties is described by the differential equations with variable coefficients. The analytical methods for solving such equations are known only for a few forms of a functions describing material properties. An overview of cases in which we can find analytical solutions to selected problems can be found in [13]. In this paper the problem for transversely isotropic half-space with Young's and shear moduli varying exponentially with depth was considered. In recent years, a number of the fundamental solutions to an exponentially graded transversely isotropic medium (half-space or layer) have been obtained [14 – 22]. The analytical solution algorithm usually consists of writing the state function in integral form, which integrands contain Fourier or Hankel transforms of the forcing factor and the fundamental solution of ordinary differential equations with variable coefficients, obtained using the integral transformations.

Parallel with the application of analytic methods for the solution of partial differential equations, a gradient isotropic coatings was also modelled by using an approach [23 – 34] according to which the coating is replaced with a package of homogeneous or inhomogeneous layers. Solution in each layer was constructed analytically. This made it possible to solve problems for functionally graded isotropic coating with arbitrarily varying properties. For transversal isotropic gradient coating, this approach was not used.

In this paper, the axisymmetric thermal conduction problem of the local heating of the surface of an inhomogeneous anisotropic half-space is investigated. The considered half space consists of a homogeneous substrate and a graded transversely isotropic coating with the slowly graded structure. In the first part of the article, the effectiveness of the algorithm based on replacing the gradient coating with continuously varying thermal properties with the package of homogeneous transversally isotropic layers is verified. For this purpose, the solution for the multilayer coating was compared with that for a transversely isotropic coating with exponentially varying properties.

In the second part of the paper, a gradient coating containing a finite number of representative cells consisting of two homogeneous isotropic layers was investigated. The considered coating was described using the homogenization method with microlocal parameters. The problem for the homogenized coating was solved using the algorithm presented in the first part of the work. Good agreement was shown between the solution for the coating described by the homogenization method and the solution for the multilayer coating with a step change in thermal properties.

2. FORMULATION OF THE PROBLEM

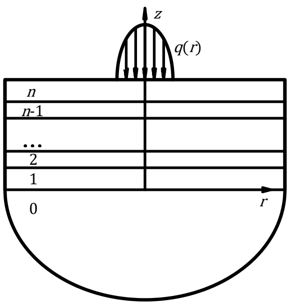

Fig. 1. The scheme of considered problem

Assume that the surface $z = h$ of the graded coated half-space is heated by a heat flux $q(r)$ on the circle of radius a, where r, z are dimensionless cylindrical coordinates referred to the linear size a, $h = H/a$, H is thickness of the coating (Fig. 1). The remaining surface of the inhomogeneous half-space is thermally insulated.

The considered half space is formed by the homogeneous substrate with the heat conductivity coefficient K_0 and a transversal isotropic gradient layer with the heat conductivity coefficients K_r and K_z, which can vary along its thickness. Let the homogeneous substrate occupies the region $0 \le r < \infty$, $-\infty < z \le 0$. The inhomogeneous coating occupies the region $0 \le r < \infty$, $0 \le z \le h$. Functions $K_r(z)$ and $K_z(z)$ are continuous in intervals (h_{i-1}, h_i), $i = 1, ..., n$, $h_0 = 0$, $h_n = h$, while points h_i, $i = 1, ..., n-1$ are discontinuity points of the first kind. On the surfaces of the discontinuity $z = h_i$, $i = 1, ..., n-1$ and on the surface $z = h_0$ between the coating and the substrate, the conditions of perfect thermal contact are satisfied.

The analysed problem reduces to a boundary problem, which involves solving the following partial differential equation

$$\frac{\partial}{\partial z}\left(K_z\frac{\partial T}{\partial z}\right) + \frac{K_r}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0, r \ge 0, z \in \cup_{i=0}^{n}(h_{i-1}, h_i) \quad (1)$$

and satisfying the boundary conditions:

$$K_z(h_n)\frac{\partial T}{\partial z}(r, h_n) = aq(r)H(1-r), \quad (2)$$

$$T(r, h_i - 0) = T(r, h_i + 0), i = 0,1, \dots, n-1, \quad (3a)$$

$$K_z(h_{i-})\frac{\partial T}{\partial z}(r, h_{i-}) = K_z(h_{i+})\frac{\partial T}{\partial z}(r, h_{i+}), i = 0,1, \dots, n-1, \quad (3b)$$

$$T(r, z) \to 0, r^2 + z^2 \to \infty, \quad (4)$$

where T is the temperature, $K_r(z) = K_z(z) = K_0$, $z < 0$, $h_{i+} = h_i + 0$, $h_{i-} = h_i - 0$, $h_{-1} \to -\infty$, $H(r)$ is Heaviside step function.

3. METHOD OF SOLUTION

The solution of the boundary value problem (1) – (4) is sought by applying the Hankel integral transformation

$$\bar{T}(s, z) = \int_0^\infty T(r, z) r J_0(sr) dr, -\infty < z \le h, s \ge 0, \quad (5)$$

where $J_0(sr)$ is the Bessel function.

Using the technique of the Hankel integral transformation, we transform the partial differential equation (1) to the form of the ordinary linear differential equation with variable coefficients

$$\frac{d}{dz}\left(K_z\frac{d\bar{T}}{dz}\right) - K_r s^2 \bar{T} = 0, z \in \cup_{i=0}^{n}(h_{i-1}, h_i). \quad (6)$$

The Hankel transform of the temperature for a homogeneous substrate that satisfies the regularity conditions at infinity (4) can be written in the form:

$$\bar{T}(s, z) = t_0(s)\exp(sz), z < 0, \quad (7)$$

where $t_0(s)$ is the unknown function.

The analytical form of the solution of the differential equation with variable coefficients (6) is known only for selected forms of the function $K_r(z)$ and $K_z(z)$.

3.1. Case A

Let $n = 1$. The dependence of the thermal conductivity coefficients on the z-coordinate is described by the formulas:

$$K_r(z) = K_r^{\text{int}}\exp(\beta z), K_z(z) = K_z^{\text{int}}\exp(\beta z), \quad (8a)$$

$$\beta = \frac{1}{h}\ln\left(\frac{K_z^{\text{sur}}}{K_z^{\text{int}}}\right), \quad (8b)$$

where K_z^{sur} is the heat conductivity coefficient on the surface of the inhomogeneous half-space in a direction normal to it.

The general solution to differential equation (6) in the Hankel transform space specified in the coating can be written in the form:

$$\bar{T} = t_1(s)\exp(\gamma_1 z) + t_2(s)\exp\big(\gamma_2(z-h)\big), 0 < z < h \quad (9)$$

where $t_1(s)$ and $t_2(s)$ are the unknown functions,

$$\gamma_{1,2} = -\frac{\beta}{2} \mp \frac{1}{2}\sqrt{\beta^2 + 4\kappa^2 s^2}, \kappa^2 = \frac{K_r^{\text{int}}}{K_z^{\text{int}}}. \quad (10)$$

Satisfying boundary conditions (2) – (4), the functions $t_i(s)$, $i = 0, 1, 2$ may be written as

$$t_i(s) = \frac{\bar{q}(s)a}{s\sqrt{K_r^{\text{sur}}K_z^{\text{sur}}}}\check{t}_i(s), \quad (11)$$

where the functions $\check{t}_i(s)$, $i = 0, 1, 2$ are obtained solution of linear equations:

$$\gamma_1\check{t}_1(s)\exp(\gamma_1 h) + \gamma_2\check{t}_2(s) = \kappa s, \quad (12a)$$

$$\gamma_1\check{t}_1(s)\exp(\gamma_1 h) + \gamma_2\check{t}_2(s) = \kappa s, \quad (12b)$$

$$K_0 s\check{t}_0(s) - \gamma_1 K_z^{\text{int}}\check{t}_1(s) - \gamma_2 K_z^{\text{int}}\check{t}_2(s)\exp(-\gamma_2 h) = 0 \quad (12c)$$

K_r^{sur} is the heat conductivity coefficient on the surface of the inhomogeneous half-space in in a plane of transverse isotropy, $\bar{q}(s)$ is Hankel transform of order 0 of function $q(r)H(1-r)$:

$$\bar{q}(s) = \int_0^1 q(r)rJ_0(sr)dr \quad (13)$$

3.2. Case B

In cases where the analytical solution of the differential equation with variable coefficients (6) is not known, the inhomogeneous coating can be replaced by a multilayer system of n homogeneous transversal isotropic layers. The thermal properties of the replacement coating are described by thermal conductivity coefficients:

$$\begin{bmatrix} K_r(z) \\ K_z(z) \end{bmatrix} = \begin{bmatrix} K_r^i \\ K_z^i \end{bmatrix}, h_{i-1} < z < h_i, i = 1,\dots,n, \quad (14)$$

where the parameters K_r^i and K_z^i are means value of functions $K_r(z)$ and $K_z(z)$ in the region (h_{i-1}, h_i) respectively.

The general solution to differential equation (6) in the Hankel transform space defined in the region (h_{i-1}, h_i), $i = 1, ..., n$ can be written in the form:

$$\bar{T} = t_{2i-1}(s)\sinh(\kappa_i^*) + t_{2i}(s)\cosh(\kappa_i^*), h_{i-1} < z < h_i \quad (15)$$

where $\kappa_i^* = s\kappa_i(h_i - z), \kappa_i^2 = K_r^i/K_z^i, i = 1, ..., n$.

Satisfying boundary conditions (2) – (3), the functions $t_i(s)$, $i = 0, 1, ..., 2n$ may be written as

$$t_i(s) = \frac{\bar{q}(s)a}{s\sqrt{K_r^n K_z^n}}\check{t}_i(s), \quad (16)$$

where the functions $\check{t}_i(s)$, $i = 0, 1, ..., 2n$ are obtained solution of linear equations:

$$\check{t}_0 - S_1\check{t}_1 - C_1\check{t}_2 = 0, \quad (17a)$$

$$K_0\check{t}_0(s) + K_1^* C_1\check{t}_1 + K_1^* S_1\check{t}_2 = 0, \quad (17b)$$

$$\check{t}_{2i-2} - S_i\check{t}_{2i-1} - C_i\check{t}_{2i} = 0, i = 2,\dots,n, \quad (17c)$$

$$K_{i-1}^*\check{t}_{2i-3} - K_i^* C_i\check{t}_{2i-1} - K_i^* S_i\check{t}_{2i} = 0, i = 2,\dots,n, \quad (17d)$$

$$\check{t}_{2n-1} = -1, \quad (17e)$$

where $S_i = \sinh(s\kappa_i(h_i - h_{i-1})), C_i = \cosh(s\kappa_i(h_i - h_{i-1}))$, $K_i^* = K_z^i\kappa_i, i = 1, \dots, n$.

3.3 Case C

Let the investigated coating in its cross section is composed of m representative cells (Fig. 2). The representative cell contains two homogeneous layers with the thermal conductivity coefficients K_1, K_2, and dimensionless thickness $\delta_1 = \eta\delta$, $\delta_2 = (1-\eta)\delta$, where $\delta = h/m$ is dimensionless cell thickness. The parameter $\eta \in (0, 1)$ describes the content of the first type material in a representative cell and can vary along the thickness of the coating.

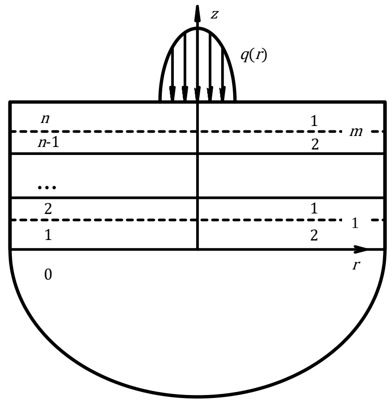

Fig. 2. The scheme of considered problem

The thermal properties of the investigated coating are described by thermal conductivity coefficients:

$$K_r(z) = K_z(z) = \begin{bmatrix} K_1, h_{2j-1} < z < h_{2j} \\ K_2, h_{2j-2} < z < h_{2j-1} \end{bmatrix}, j = 1,\dots,m. \quad (18)$$

The temperature induced in the considered inhomogeneous half-space can be evaluated through the use of two different concepts. One concept implies the analysis of the actual strata through writing the solution in the form of the formulas (15) and solving the system of equations (17) in which: $n = 2m$, $\kappa_i = 1$, $i = 1, ..., n$, $K_z^{2j} = K_1$, $K_z^{2j-1} = K_2$, $j = 1, ..., m$.

3.4 Case D

A large number boundary conditions on the interfaces complicates the solution of the problem. Another approach is using a homogenized model in which properties of the homogenized coating are determined on the base of properties of the components. In the latter case, the material properties across the coating depend on the continuous function $\eta(z)$ chosen so that the relationships are satisfied:

$$\eta_j = \frac{1}{h_{2j} - h_{2j-2}}\int_{h_{2j-2}}^{h_{2j}} \eta(z)dz, j = 1,\dots,m, \quad (19)$$

where the parameter η_j describes the volume fraction of the layer of the first kind in the representation cell with the number j.

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According to the homogenization method with microlocal parameters [7, 8], the solution in the coating is represented by the temperature $T^{\mathrm{hom}}(r,z)$ which is determined by the thermal properties of the two layers forming the representative cell with account for the function $\eta(z)$. The function $T^{\mathrm{hom}}(r,z)$ is obtained from solving boundary problem (1) – (4), with $n=1$ and the heat conductivity coefficients computed through the use of the homogenization method in the form, as follows:

$$K_r(z) = K_1\eta(z) + K_2\left(1-\eta(z)\right), 0<z<h, \tag{20a}$$

$$K_z(z) = \frac{K_1K_2}{K_2\eta(z)+K_1\left(1-\eta(z)\right)}, 0<z<h\,. \tag{20b}$$

It should be noted that solving the problem with the homogenization method leads to the differential equation with variable coefficients (6), for which the analytical solution is not known. Therefore, the homogenized coating is replaced by a multilayer array of n homogeneous transverse isotropic layers. As in the section B, the thermal properties of the layers are described by formulas (14).

As a result of solving the system of equations (12) in case A or the system of equations (17) in cases B, C or D, the Hankel transform of the temperature is found. In order to restore the found temperature in the physical domain, we employ the inverse transform:

$$T(r,z) = \int_0^{\infty} \bar{T}(s,z) s J_0(sr)ds\,, -\infty<z\le h, r\ge 0\,. \tag{21}$$

The integral (21) at internal points of the considered inhomogeneous half space ($z<h$) is taken with the help of the Gaussian quadrature. On the surface $z=h$, the considered integral is computed with regard for the asymptotic behavior of the function $\bar{T}(s,h)$ as $s\to\infty$:

$$\lim_{s\to\infty}\bar{T}(s,h) = \frac{a\bar{q}(s)}{s\sqrt{K_r(h)K_z(h)}}. \tag{22}$$

The integral in which the integrand is replaced by its asymptotic is taken analytically. It should be noted that the formula (22) describes the Hankel transform of order 0 of the temperature in a homogeneous transversal isotropic half-space with the heat conductivity coefficients $K_r(h)$ and $K_z(h)$. To find the remaining integrals, we apply the Gaussian quadrature.

4. NUMERICAL EXAMPLES AND DISCUSSION

We assume that

$$q(r) = q_0\sqrt{1-r^2}, r<1. \tag{23}$$

The Hankel transform of the function $q(r)H(1-r)$ is described by equation

$$\bar{q}(s) = q_0\sqrt{\frac{\pi}{2}}\frac{J_{3/2}(s)}{s\sqrt{s}}, \tag{24}$$

where $J_{3/2}(s)$ is the Bessel function.

Formula to calculate the temperature over the surface of the homogeneous isotropic half-space with the heat conductivity coefficient K_0 have form

$$\frac{12TK_0}{aq_0} = \begin{cases}3\pi(1-0.5r^2), r\le 1,\\ 4r^{-1}F(0.5,0.5;2.5;r^{-2}), r>1,\end{cases} \tag{25}$$

where F is hypergeometric function.

4.1 Case AB

First, we intend to verify the effectiveness of the algorithm based on replacing the gradient coating with continuously varying thermal properties with the package of homogeneous transversal isotropic layers. To this end, the solution for a gradient coating whose thermal properties are described by formulas (8) will be obtained using two approaches. The first approach is described in the section A, and the second approach is described in the section B.

If the functions $K_r(z)$ and $K_z(z)$ are described by formula (8), then:

$$\begin{bmatrix}K_r^i\\K_z^i\end{bmatrix} = \begin{bmatrix}K_r^{\mathrm{int}}\\K_z^{\mathrm{int}}\end{bmatrix}\frac{\exp(\beta h_i)-\exp(\beta h_{i-1})}{\beta(h_i-h_{i-1})}, \tag{26a}$$

$$\kappa_i^2 = \kappa^2 = \frac{K_r^{\mathrm{int}}}{K_z^{\mathrm{int}}}, i=1,\ldots,n\,. \tag{26b}$$

The analysis of the obtained relations enables us to conclude that the solution of the posed problem of modelling of the gradient coating by a package of layers depends on five dimensionless parameters: the ratios of the heat conductivity coefficients K_z^{sur}/K_0, K_z^{int}/K_0 and K_r^{int}/K_0, the dimensionless thickness of the coating h and the number of layers in the package n. The solution of the problem for the inhomogeneous coating obtained with regard for the continuous dependence of the thermal properties depends on the first four indicated parameters. In order to decrease the number of parameters, it will be assumed that the gradient coating is a thermal insulator with the heat conductivity coefficients: $K_z^{\mathrm{sur}}=0.2K_0$, $K_z^{\mathrm{int}}=K_0$, $K_r^{\mathrm{int}}=K_0$ or $5K_0$. In addition, it is assumed that: $h=0.4$ and $n=10, 20, 40, 80$ and 160.

Figure 3 show the distributions of the temperature over the surface of the inhomogeneous half-space $z=h$ (black lines and rhombi) and over the interface between coating and base $z=0$ (grey lines and rhombi) for two values of the parameter κ^2.

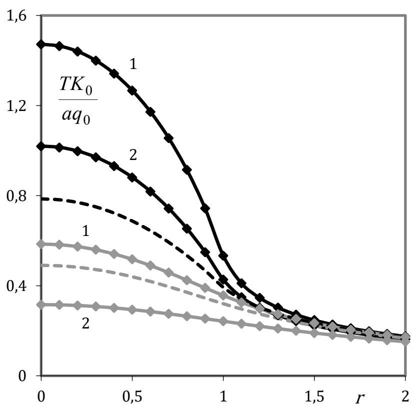

Fig. 3. Distributions of the temperature over the surface z = h (black lines and rhombi) and over z = 0 (grey lines and rhombi), dashed lines – the distributions of the temperature for the homogeneous half-space: 1 – κ2 = 1; 2 – κ2 = 5; n = 20

Distributions of the radial heat flux along the surface $z=h$ are shown in Figure 4. In this figures, the continuous lines correspond to the solution of the problem with continuous variation of the thermal properties. The rhombi correspond to the results obtained for a package formed by 20 homogeneous transversal isotropic layers.

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The dashed lines in Figures 3 and 4 describe the solution for a isotropic homogeneous half-space with the heat conductivity coefficient K_0. The results of calculations presented in Figs. 3 and 4 show good agreement between the solutions obtained using the analysed two models of the considered coating.

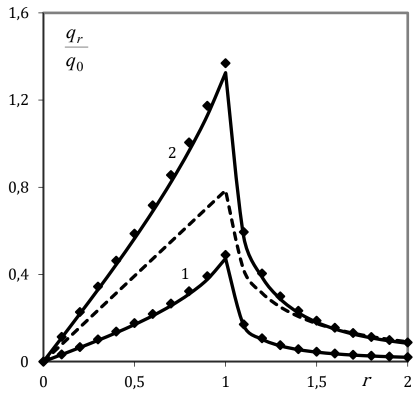

Fig. 4. Distributions of the radial heat flux along the surface $z = h$, dashed line – the distributions of the radial heat flux for the homogeneous half-space: 1 – $\kappa^2 = 1$; 2 – $\kappa^2 = 5$; $n = 20$.

As was to be expected, the highest temperature value is observed in the middle of the heating zone. At the interface between the coating and the substrate the maximum temperature occurs at the point $r = z = 0$. The maximum radial heat flux is formed on the boundary of the heating zone.

Tab. 1. Dependence of the dimensionless parameters $T_{\text{sur}}^{\max} = \frac{T(0,h)K_0}{aq_0}$, $T_{\text{int}}^{\max} = \frac{T(0,0)K_0}{aq_0}$, $q_{r,\text{sur}}^{\max} = \frac{q_r(1,h)}{q_0}$ on the dimensionless parameter κ^2 and number of the layers n

κ^2	n	$T_{\text{sur}}^{\max}$	$\varepsilon_{T_{\text{sur}}}$,%	$T_{\text{int}}^{\max}$	$\varepsilon_{T_{\text{int}}}$,%	$q_{r,\text{sur}}^{\max}$	$\varepsilon_{q_{\text{sur}}}$,%
1	∞	1.4723		0.5847		0.4733	
	160		-0.0011		-0.0002		0.5103
	80		-0.0036		-0.0007		0.9445
	40		-0.0135		-0.0028		1.7952
	20		-0.0537		-0.0113		3.4317
	10		-0.2146		-0.0451		6.5039
5	∞	1.0200		0.3158		13256	
	160		-0.0017		-0.0007		0.4731
	80		-0.0064		-0.0017		0.9078
	40		-0.0252		-0.0054		1.7408
	20		-0.1005		-0.0204		3.2928
	10		-0.4007		-0.0799		6.0922

The dimensionless values $T(0,h)K_0/a/q_0$, $T(0,0)K_0/a/q_0$ and $q_r(1,h)/q_0$ for the problem with continuous variation of thermal properties for two values of parameter κ^2, are presented in the corresponding columns of Table 1. In order to compare the difference between solutions, which is caused by an application of the two proposed approaches to solving the problem in the rows with $n = 160, 80, 40, 20$ and 10 we present the relative deviations (given in percent's) obtained for the multi-layered coating with the indicated number of the layers. As can be seen from Table 1, the double increase in the layer number cause the fourfold decrease in the difference between the analyzed temperature values and the twofold decrease in the difference between the analyzed radial heat flux. The good agreement in temperature calculation is observed even for a relatively small number of layers in the package $n = 10$. The greatest difference between the analyzed solutions when calculating the radial heat flux at the boundary of the heating zone. It should be noted that the relative big difference occurs only in a small area around the point $r = 1$, $z = 0$. For 40 layers in the package this difference do not exceed 2%.

Lines 1 in Figures 3 and 4 describe an insulating coating with isotropic properties, i.e., a coating with equal heat conductivity coefficients in the plane $z = \text{const.}$ and in a direction normal to it. As can be seen from Figure 3, the temperature level in the heating zone in the half-space with such coating is much higher than the corresponding temperature level in the half-space without coating. The temperature level on the surface of the substrate is also higher. This means that in the problems, in which the quantity of the heat generated in the heating zone is the input parameter (for example in the problems with the heat generated by friction), an insulating coating does not only necessarily decrease the temperature level in the base, but even increases it. From the Figure 3 also it is shown that for the coating with the ratio of the heat conductivity coefficients $\kappa^2 = 5$ we observe a significant decrease of the temperature in the heating zone and on the interface between the coating and the base. It is caused by the increase of the radial heat flux (line 2 in Figure 4).

4.2 Case CD

In the section AB a hypothetical gradient covering whose thermal properties are described by formulas (8) was investigated. A coating structure that would make it possible to obtain a coating with assumed thermal properties was not considered. In this section the multilayer coating described in section C is analyzed.

The analysis of relations shows, the temperature in the homogenized model (section D) depends on the function $\eta(z)$ and the three dimensionless parameters: K_1/K_0, K_2/K_0 and h. However if the coating is treated as a multilayered body (section C), one should take into account the number of a representative cells m.

To simplify, let us assume that the content of the first type material in the representative cell is described by a linear function of the coordinate z, i.e.:

$$\eta(z) = \frac{z}{h}, 0 < z < h \tag{27}$$

The parameters describing the content of the first type material in the representative cells numbered $i = 1, \ldots, m$ are calculated using the following formulas

$$\eta_i = \frac{2i-1}{2m}, i = 1, \ldots, m\,. \tag{28}$$

The coating described by the homogenization method is replaced by a multilayer coating consisting of N homogeneous transversal isotropic layers of equal thickness $h^* = h/N$. The thermal properties of layers are described by formulas:

$$K_r^l = K_2 + (K_1 - K_2)\frac{2l-1}{2N}, l = 1, \ldots, N, \tag{29a}$$

$$K_z^l = \frac{K_1K_2N}{K_2-K_1}\ln\frac{K_1N+(K_2-K_1)l}{K_1N+(K_2-K_1)(l-1)}, l = 1, \ldots, N. \tag{29b}$$

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As in the section AB, the gradient coating is a thermal insulator: $K_1/K_0 = 0.2$; $K_2/K_0 = 1$ or 5. In addition, it is assumed that: $h = 0.4$ and $m = 5, 10, 20, 40$ and 80.

Based on the comparison of the temperature values in the homogenized coating calculated for different values of the parameter N, it can be concluded that the calculation error will not exceed 0.2% when the homogenized coating is replaced by a package of 20 homogeneous transversal isotropic layers with a slowly changing thermal properties. If $N = 10$, the error of temperature calculation does not exceed 0.5%.

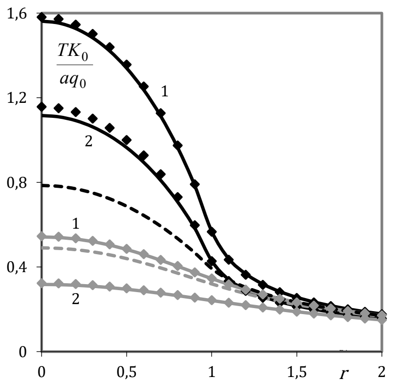

Fig. 5. Distributions of the temperature over the plane z = h (black lines and rhombi) and over the plane z = 0 (grey lines and rhombi), dashed lines – the distributions of the temperature for the isotropic homogeneous half-space with the heat conductivity coefficient K0: 1 – K2/K0 = 1; 2 – K2/K0 = 5 ; m = 10

In the following analysis, we will focus on comparing solutions based on the two mathematical models of the considered coating described in sections C and D. Figure 5 shows the distributions of the temperature over the surface of the inhomogeneous half-space (black lines and rhombi) and over the interface between coating and base (grey lines and rhombi). The continuous lines describe the distributions of the temperature obtained within the framework of the homogenization method described in section D. The rhombi mark the numerical results obtained for the multilayer coating with a step change in thermal properties on the surfaces between the layers (the approach described in section C). The dashed lines describe the solution in the isotropic homogeneous half-space with the heat conductivity coefficient K_0.

Tab. 2. Dependence of the dimensionless parameters $T_{\text{sur}}^{\max} = \frac{T(0,h)K_0}{aq_0}$, $T_{\text{int}}^{\max} = \frac{T(0,0)K_0}{aq_0}$ on the dimensionless parameter K_2/K_0 and number of the representative cells m

K_2/K_0	m	$T_{\text{sur}}^{\max}$	$\varepsilon_{T_{\text{sur}}}$,%	$T_{\text{int}}^{\max}$	$\varepsilon_{T_{\text{int}}}$,%
1	∞	1.5630		0.5420	
	80		0.1613		0.0992
	40		0.3173		0.1965
	20		0.6171		0.3854
	10		1.1698		0.7407
	5		2.1096		1.3661
5	∞	1.1164		0.3217	
	80		0.4999		0.2566
	40		0.9858		0.5106
	20		1.9225		0.9844
	10		3.6628		2.0010
	5		6.6678		3.8612

The results of calculations presented in Fig. 5 show good agreement between the distributions of the temperature obtained using the analyzed two models of the considered coating. We can estimate the differences obtained when calculating the maximum temperatures in the planes $z = h$ and $z = 0$ from the calculation results shown in Table 2. The maximum temperatures for the problem described in section D are presented in the columns $T_{\text{sur}}^{\max}$ and $T_{\text{int}}^{\max}$ of Table 2. The relative deviations (given as percentages) that result from the two proposed approaches to solving the problem are shown in the rows with $m = 80, 40, 20, 10$ and 5. As can be seen from Table 2, the double increase in the representative cells number in the multilayer coating with a step change in thermal properties cause the double decrease in the deviations between the analyzed temperature values. Larger differences between temperature values occur when one layer in the representative cell is a heat conductor. If $m = 20$, the error in calculating the parameters $T_{\text{sur}}^{\max}$ and $T_{\text{int}}^{\max}$ does not exceed 2% and 1% respectively.

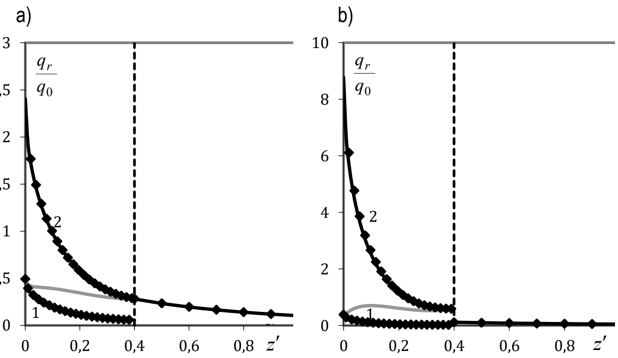

Fig. 6. Distributions of the radial heat flux on the cylindrical surface $r = 1$ (black lines and rhombi): fig. a) – $K_2/K_0 = 1$; fig. b) – $K_2/K_0 = 5$; grey line – the distribution of the average radial heat flux; vertical dashed line – the interface of the coating and base; $m = 20$; $z' = h - z$

The temperature in the considered problem can be treated as a macro-characteristic, which does not depend on the choice of component of the representative cell. An example of micro-characteristic is the radial heat flux. Figures 6 presents the distributions of the radial heat flux on the cylindrical surface $r = 1$ containing the edge of the heating zone. When the homogenization method is applied, there is no information connected with the kind of the layer in the specified point of coating. At each point of the coating we obtain two equations to calculate the radial heat flux

$$q_r^{(l)}(r,z) = -K_l \frac{\partial T^{\text{hom}}(r,z)}{\partial z}. \qquad (30)$$

The equation with the index 1 allows to determine of the radial heat flux in the layer of the first kind, and the one with the index 2 – in the layer of the second kind. Two continuous lines denoted by numbers 1 and 2 (the indexes of types of layers) are appropriate for the values of radial heat flux in the layer of the representative cell numbered 1 or 2, respectively. The rhombi are adequate for the problem for the multilayer coating with a step change in thermal properties (section C). Averaged within the representational cell, the radial heat flux values calculated according to the formula

$$q_r^*(r,z) = -K_r(z)\frac{\partial T^{\text{hom}}(r,z)}{\partial z} \quad (31)$$

are described in the figure with a gray line. The vertical dashed line indicate the interface of the coating and base.

If the radial heat flux is calculated in the layers with even numbers, the adequate rhombi are consistent with the continuous line denoted by 1, in the layers with odd numbers, with the continuous line denoted by 2. This means that the continuous lines within the homogenized model correctly determine the distribution of radial heat flux in the both layers of the representative cell.

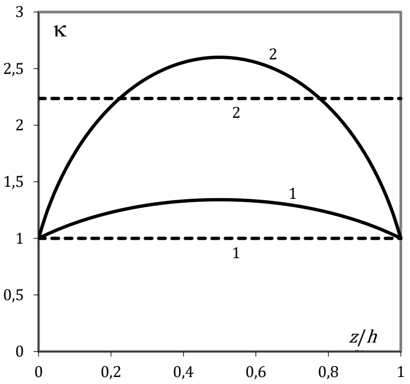

Fig. 7. Graphs of the function $\kappa(z)$: continuous lines – coatings described in section C: 1 – $K_1/K_0 = 0.2$, $K_2/K_0 = 1$; 2 – $K_1/K_0 = 0.2$, $K_2/K_0 = 5$; dashed lines – coatings described in section A: 1 – $\kappa^2 = 1$; 2 – $\kappa^2 = 5$

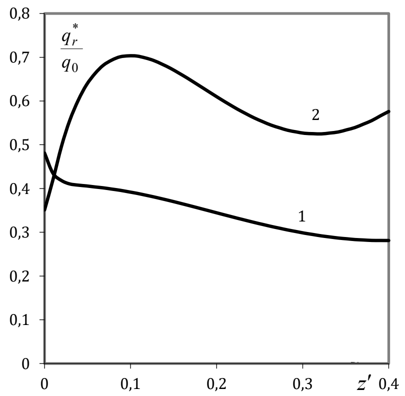

Fig. 8. Distributions of the average radial heat flux on the cylindrical surface $r = 1$ (coatings described in section C): 1 – $K_1/K_0 = 0.2$, $K_2/K_0 = 1$; 2 – $K_1/K_0 = 0.2$, $K_2/K_0 = 5$; $m = 20$; $z' = h - z$

Comparing Figures 3 and 5, we conclude that the temperature distributions in both figures are very similar. The multilayer coating with thermal conductivity coefficients $K_1 = 0.2K_0$ and $K_2 = K_0$ is not an isotropic covering. However, Figure 7 shows that the values of the function $\kappa(z)$ for this coating only slightly exceed the value of $\kappa = 1$, which describes a gradient isotropic coating. The average value of this function is about 1.238. Therefore, the curves numbered 1 in Figures 3 and 5 differ only slightly in these figures. Curve 2 in Figure 7 is a graph of the function $\kappa(z)$, which describes a multilayer coating whose representative cell consists of a thermal insulator with thermal conductivity coefficient $K_1 = 0.2K_0$ and a heat conductor with thermal conductivity coefficient $K_2 = 5K_0$. The average value of this function is 2.2, and the second coating considered in section AB is described by a constant parameter $\kappa = \sqrt{5} \approx 2.236$. This shows that there is a high correlation between the temperature distribution and the average value of the function $\kappa(z)$.

As the average value of the function $\kappa(z)$ increases, it increases the average radial heat flux (Fig. 8). In the gradient coating considered in the CD section, this increases heat collection along layers that are heat conductors in representative cells, and their heat conduction coefficient is higher than of the base heat conduction coefficient. This results in lower temperatures in both the heating zone and the base.

5. CONCLUSIONS

In the paper the method of solving heat conduction problems for coated solids is presented. The considered body is modeled by a inhomogeneous half-space consisting of a homogeneous substrate and a transversal isotropic coating with arbitrary variations of thermal properties along the thickness. The investigated coating is replaced by a package of homogeneous transversal isotropic layers.

To verify the method, the solution for an appropriately selected multilayer coating was compared with the analytical solution of the problem in which the thermal properties of the coating are described by exponential functions. A comparison of the obtained solutions showed that a twofold increase in the number of layers cause a fourfold decrease in the difference between the analyzed temperature and a twofold decrease in the difference between the analyzed radial heat flux. Sufficient agreement for engineering applications in temperature calculations was observed with 10 layers in the package. The conducted tests allow us to suggest that the proposed method is effective for the transversely isotropic gradient coatings, whose the thermal properties are described by arbitrary piecewise continuous functions of the distance to the coating surface.

In the second part of the paper, the proposed solution method was used to analyze a multilayer coating with a step change of the thermal properties. The tested coating consisted of a finite number of a representative cells containing two homogeneous isotropic layers with different thermal conductivity coefficients. The content of the first type material in a representative cell was allowed to vary along the thickness of the coating.

For a large number of layers, the temperature and heat flux induced in the considered inhomogeneous half-space can be calculated using two different concepts. One concept implies the analysis of the actual layers. The other one is concerned with homogenization procedure leading to a FGM-coating with continuously varying thermal properties.

Previous studies [9, 10] have shown that the homogenization is effective when the content of the first type material in a representative cell is constant, i.e. the investigated coating has a periodic structure. Similar studies for a gradient coatings with slowly varying properties have been carried out only for one-dimensional problems [11, 12].

The main difficulty of the present study compared to the study of the coating with the periodic structure is that the homogenized gradient coating is described by differential equations, for which the analytical solution is not known. Therefore, the algorithm for solving the problem for the obtained homogenized coating uses the approach described in the first part of the article. This means that in

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the conducted research two packages of layers are compared. The first package contains layers with step changes in thermal properties. The second consists of layers with slowly changing properties. The comparison of solutions has shown that it is possible to select the second package of layers in such a way that the difference between the obtained solutions is negligible. The main advantage of the second package of layers is that satisfactory results for engineering applications are obtained for 10 layers in the package.

It is also shown that the proposed homogenization approach allows to correctly calculate not only the temperature, which is the average characteristic in a representative cell, but also the radial heat flux, the value of which depends on the choice of the component in the representative cell.

It should be emphasized that the presented research suggests that it will allow to correctly solve a more complicated problems, in particular axisymmetric and three-dimensional thermoelasticity problems.

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